Row Scaling As A Preconditioner For Certain Nonsymmetric Linear Systems With Discontinuous Coefficients

D. Gordon
Department of Computer Science, University of Haifa, Haifa 31905, Israel
gordon@cs.haifa.ac.il
R. Gordon
Department of Aerospace Engineering, The TechnionIsrael Inst. of Technology, Haifa 32000, Israel
rgordon@tx.technion.ac.il

Linear systems with large differences between coefficients, called "discontinuous coefficients", arise in many cases in which partial differential equations (PDEs) model physical phenomena involving heterogeneous media. The standard approach to solving such problems is to use domain decomposition (DD) techniques, with domain boundaries conforming to the boundaries between the different media. This approach can be difficult to implement when the geometry of the domain boundaries is complicated or the grid is unstructured. This work examines the simple preconditioning technique of scaling the equations by dividing each equation by the L_p-norm of its coefficients. This preconditioning is called geometric scaling (GS). GS is a particular form of a diagonal preconditioner. In the literature, diagonal scaling is usually applied to both sides of the system matrix in order to preserve symmetry and enable the use of the conjugate gradient (CG). This work is restricted to nonsymmetric systems.

It has been long been known that diagonal scaling improves the condition number of the system matrix and the convergence properties of some algorithms; see [9, 11]. Gambolati et al. [2] use the least square logarithm (LSL) scaling on the rows and the columns of the system matrix for a certain problem in geomechanics with discontinuous coefficients.

However, it seems that there is no study of the general usefulness of row scaling for discontinuous coefficients. We examine several nonsymmetric problems derived from PDEs with discontinuous coefficients and small to moderate convection terms. It is shown that GS improves the convergence properties of some solution methods applied to these problems. The solution methods that we tested are restarted GMRES and Bi-CGSTAB, with and without the ILUT preconditioner. These four algorithm/preconditioner combinations were tested on both the original and the scaled systems, and it is shown that GS improves the convergence properties of these methods.

Tests were done on the following four nonsymmetric problems:

(1) Problem F2DB from Saad [7, §3.7].
(2) Example 2 from van der Vorst [10], to which we added a convection term.
(3) Example 4 from van der Vorst [10].
(4) A problem from Graham and Hagger [6], to which we added a convection term.

Problems 2 and 4 were originally symmetric, but the extra convection term turned them into nonsymmetric problems.

Sample results are shown for Problem 1 in Table 1. Three relative residual (rel-res) criteria were prescribed, and both the time and the number of iterations to reach the goal are shown. In cases of stagnation, the table shows the relative residual achieved. GMRES was restarted after ten iterations, and ILUT was used with drop tolerance = 0 and fill-in = 1. All tests were done with the AZTEC package [8], in which GMRES is implemented with a double classical Gram-Schmidt orthogonalization step. GS was used with the L_2-norm, but the L_1-norm produced similar results.

Table 1: No. of iterations and runtimes for Problem 1. Grid size = 128×128.

	No. of iterations and time (in sec.)		
Method	rel-res = 10^{-4}	rel-res = 10^{-7}	rel-res = 10^{-10}
Bi-CGSTAB	no conv.	no conv.	no conv.
with GS	91 (0.30)	299 (0.99)	361 (1.19)
Bi-CGSTAB+ILUT	31 (0.23)	107 (0.67)	142 (0.88)
with GS	30 (0.23)	90 (0.59)	130 (0.81)
GMRES(10)	converged to 3.8×10^{-2}		
with GS	265 (0.85)	converged to 1.1×10^{-5}	
GMRES(10)+ILUT	converged to 3.9×10^{-3}		
with GS	39 (0.23)	converged to 1.1×10^{-5}	

The effects of GS can be summarized as follows: in most cases, when the tested method converges to the specified accuracy criterion, GS speeds up the convergence. In many cases, when the tested method stagnates on the original system, it converges on the scaled system. When GMRES (with and without ILUT) stagnates before reaching the prescribed convergence goal, GS postpones the stage at which stagnation sets in, and enables convergence to a level that is acceptable for most practical applications.

These results do not imply that GS is the "best" preconditioner for these problems, or that it competes with DD methods in terms of runtime efficiency. Every particular problem has its own specific most efficient algorithm/preconditioner/DD combination. However, the results indicate that GS is a simple, generally useful preconditioner for discontinuous coefficients. In some practical situations, it may save the search for a complicated DD method and/or algorithm/preconditioner. Another advantage of GS is that it is inherently parallel.

The effect of GS on the distribution of the eigenvalues was also studied. It is generally accepted that a large accumulation of eigenvalues near the origin is detrimental to convergence. GS "pushes" many eigenvalues away from the origin. When the eigenvalue range is divided into 100 equal intervals, GS reduces the number of eigenvalues in the first interval (percentile) by one to three orders of magnitude.

GS is useful when the convection terms are small to moderate. It was shown in previous work that for strongly convection-dominated systems, Björck and Elfving's CGMN algorithm [1, 5] and its block-parallel version CARP-CG [4] (which is a CG-acceleration of CARP [3]) provide useful solution methods. It will shown in future work that these algorithms also perform well when, in addition to large convection terms, the coefficients are discontinuous. This is probably due to the fact that GS, with the L_2-norm, in inherent in these algorithms. Note that these methods also converge with small to moderate convection terms, but they are less efficient than the Bi-CGSTAB/GMRES/ILUT (with GS) methods.

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