

# Efficient dropping technique for ILU and IUL factorizations obtained as by-products of FFAPINV and BFAPINV algorithms

List of authors:

A. Rafei<sup>1</sup>

In this paper, we propose an efficient dropping criterion for IUL factorization obtained from Backward Factored APproximate INverse (BFAPINV) and ILU factorization obtained from Forward Factored APproximate INverse (FFAPINV) algorithms.

Suppose that matrix  $A$  is nonsymmetric. Also suppose that  $L$  is an unit lower triangular matrix and  $U$  is an upper triangular matrix. In [2], an ILU factorization of matrix  $A$ , which is obtained as by-product of FFAPINV [3] process, has been presented.

We term this ILU factorization, ILUFF (ILU factorization obtained from Forward Factored APproximate INverse). The following algorithm computes the ILUFF preconditioner.

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## Algorithm 1 (ILUFF factorization)

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1.  $w_1 = e_1^T$ ,  $z_1 = e_1$ ,  $d_1 = a_{11}$ .
  2. **for**  $j = 2$  to  $n$  **do**
  3.  $w_j = e_j^T$ ,  $z_j = e_j$ .
  4. **for**  $i = 1$  to  $j - 1$  **do**
  5.  $L_{ji} = \frac{A_{j,:}z_i}{d_i}$   $U_{ij} = \frac{w_i A_{:,j}}{d_i}$
  6. apply a dropping rule to  $L_{ji}$  and to  $U_{ij}$
  7.  $z_j = z_j - (\frac{w_i A_{:,j}}{d_i})z_i$ ,  $w_j = w_j - (\frac{A_{j,:}z_i}{d_i})w_i$
  8. for all  $l \leq i$  apply a dropping rule to  $z_{lj}$  and to  $w_{lj}$
  9. **end for**
  10.  $d_j = w_j A_{:,j}$  {not positive definite}
  11.  $d_j = w_j A w_j^T$  {positive definite}
  12. **end for**
  13. Return  $L = (L_{ij})$  and  $U = (d_i U_{ij})$ .
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Two below dropping strategies can be used to drop entries of  $z_j$  and  $w_j$  vectors in Algorithm 1.

- **First strategy:** At each step  $j$  of Algorithm 1, entries  $z_{lj}$  and  $w_{jl}$  for  $l \leq i$  are dropped when

$$|z_{lj}| \leq \varepsilon_Z, \quad |w_{jl}| \leq \varepsilon_W. \quad (1)$$

- **Second strategy:** At each step  $j$  of Algorithm 1, the whole vectors

$$z_j = e_j - \sum_{i=1}^{j-1} \left( \frac{w_i A_{:,j}}{d_i} \right) z_i, \quad w_j = e_j^T - \sum_{i=1}^{j-1} \left( \frac{A_{j,:} z_i}{d_i} \right) w_i,$$

will be computed and then, entries  $z_{lj}$  and  $w_{jl}$ , for  $l \leq j$ , that satisfy the former dropping criterion (1) will be dropped.

The following proposition presents an efficient way of dropping to compute the ILUFF preconditioner [1]. In propositions 1 and 2, we consider  $Z_{:,k}$  as the  $k$ -th column of matrix  $Z$  and  $W_{k,:}$  as the  $k$ -th row of matrix  $W$ .

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<sup>1</sup>Department of Applied mathematics, Sabzevar Tarbiat Moallem University, Sabzevar, Iran. Email: rafei@sttu.ac.ir, rafei.am@gmail.com.

**Proposition 1.** Let  $\varepsilon_{U,Z}$  and  $\varepsilon_{L,W}$  be the same drop tolerance parameter for  $U, Z$  and  $L, W$  matrices, respectively. Suppose that at each step  $j$  of Algorithm 1, entries  $L_{jk}$  and  $U_{kj}$ , for  $k < j$ , be dropped when the criterions

$$|L_{jk}| \|W_{k,:}\|_1 \leq \varepsilon_{L,W}, \quad |U_{kj}| \|Z_{:,k}\|_\infty \leq \varepsilon_{U,Z}, \quad (2)$$

satisfied. Thus, for  $1 \leq i \leq j \leq n$

- if the first dropping strategy be applied to drop entries of  $z_j$  and  $w_j$  vectors in each step  $j$ , then

$$|(I - ZU)_{ij}| \leq 2(j - i)\varepsilon_{U,Z}, \quad |(I - LW)_{ji}| \leq 2(j - i)\varepsilon_{L,W}.$$

- if the second dropping strategy be applied to drop entries of  $z_j$  and  $w_j$  vectors in each step  $j$ , then

$$|(I - ZU)_{ij}| \leq (j - i + 1)\varepsilon_{U,Z}, \quad |(I - LW)_{ji}| \leq (j - i + 1)\varepsilon_{L,W}.$$

Suppose that  $U$  is an unit upper triangular matrix. Consider  $L$  as a lower triangular matrix. The following algorithm computes the incomplete IUL factorization

$$A \approx UL,$$

which is obtained as by-product of BFAPINV [4] process. We term this IUL factorization as IULBF (IUL factorization obtained from Backward Factored APproximate INverse).

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**Algorithm 2 (IULBF factorization)**

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1.  $w_n = e_n^T, \quad z_n = e_n, \quad d_n = a_{nn}.$
  2. **for**  $j = n - 1$  to  $1$  **do**
  3.    $w_j = e_j^T, \quad z_j = e_j.$
  4.   **for**  $i = j + 1$  to  $n$  **do**
  5.      $U_{ji} = \frac{A_{j:,i}}{d_i}, \quad L_{ij} = \frac{w_i A_{:,j}}{d_i}$
  6.     apply a dropping rule to  $U_{ji}$  and to  $L_{ij}$
  7.      $z_j = z_j - (\frac{w_i A_{:,j}}{d_i})z_i, \quad w_j = w_j - (\frac{A_{j:,z_i}}{d_i})w_i$
  8.     for all  $l \geq i$  apply a dropping rule to  $z_{lj}$  and to  $w_{jl}$
  9.   **end for**
  10.    $d_j = w_j A_{:,j}$  {not positive definite}
  11.    $d_j = w_j A w_j^T$  {positive definite}
  12. **end for**
  13. Return  $L = (d_i L_{ij})$  and  $U = (U_{ij}).$
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Two below dropping strategies can be used to drop entries of vectors  $z_j$  and  $w_j$  in Algorithm 2.

- **First strategy:** At each step  $j$  of Algorithm 2, entries  $z_{lj}$  and  $w_{jl}$ , for  $l \geq i$ , are dropped when criterion (1) is satisfied.
- **Second strategy:** At each step  $j$  of Algorithm 2, the whole vectors

$$z_j = e_j - \sum_{i=j+1}^n \left( \frac{w_i A_{:,j}}{d_i} \right) z_i, \quad w_j = e_j^T - \sum_{i=j+1}^n \left( \frac{A_{j:,z_i}}{d_i} \right) w_i,$$

will be computed and then, entries  $z_{lj}$  and  $w_{jl}$ , for  $l \geq j$ , that satisfy the dropping criterion (1) will be dropped.

The following proposition presents an efficient way of dropping to compute the IULBF preconditioner [1].

**Proposition 2.** *Let  $\varepsilon_{U,W}$  and  $\varepsilon_{L,Z}$  be the same drop tolerance parameter for  $U, W$  and  $L, Z$  matrices, respectively. Suppose that at each step  $j$  of Algorithm 2, entries  $L_{kj}$  and  $U_{jk}$ , for  $k > j$ , are dropped when the criterions*

$$|L_{kj}| \|Z_{:,k}\|_\infty \leq \varepsilon_{L,Z}, \quad |U_{jk}| \|W_{k,:}\|_1 \leq \varepsilon_{U,W}. \quad (3)$$

satisfied. Thus, for  $1 \leq j \leq i \leq n$

- if the first dropping strategy be applied to drop entries of  $z_j$  and  $w_j$  vectors in each step  $j$ , then

$$|(I - UW)_{ji}| \leq 2(i - j)\varepsilon_{U,W}, \quad |(I - ZL)_{ij}| \leq 2(i - j)\varepsilon_{L,Z}.$$

- if the second dropping strategy be applied to drop entries of  $z_j$  and  $w_j$  vectors in each step  $j$ , then

$$|(I - UW)_{ji}| \leq (i - j + 1)\varepsilon_{U,W}, \quad |(I - ZL)_{ij}| \leq (i - j + 1)\varepsilon_{L,Z}.$$

## References

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