A Helmholtz Iterative Solver with No Use of Finite-Difference Approximations

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1. The numerical solution of the Helmholtz equation in 3D heterogeneous media is of great importance for theory and practice of seismic waves' propagation. The common approach to its solution lies in the finite-difference approximation with subsequent application of some iterative techniques for solving a system of linear algebraic equations ([1], [2]). One of the main trouble on this way is indefiniteness of the matrix which seriously hampers convergence of iterations. Therefore, the development of effective preconditioners is crucial here. In this work, we construct the right preconditioner as an inverse of the Helmholtz operator for vertically heterogeneous background with complex refractive index.

2. Let us consider Helmholtz equation with 3D heterogeneous wave propagation velocity $c(\vec{x})$:

$$L < u >:= \left(\Delta + \frac{\omega^2}{c^2(x)}\right) u(\vec{x};\vec{x}_s;\omega) = f(\vec{x}_s;\omega) \tag{1}$$

satisfying the vanishing attenuation principle ([4]):

$$u(\vec{x};\omega) = \lim_{\varepsilon\to 0} u(\vec{x};\omega + i\varepsilon);$$
$$u(\vec{x};\omega + i\varepsilon) \to 0 \ for \ \sqrt{x^2+y^2+z^2} \to \infty$$

Let us introduce now another Helmholtz operator:

$$L_0 := \Delta + (1+i\beta)\frac{\omega^2}{c_0^2(z)}$$

with function $c_0(z)$ and attenuation β chosen in a specific way, and decompose the 3D heterogeneous Helmholtz operator L as follows:

$$L = L_0 - \delta L = \Delta + (1+i\beta)\frac{\omega^2}{c_0^2(z)} - \left((1+i\beta)\frac{\omega^2}{c_0^2(z)} - \frac{\omega^2}{c^2(\vec{x})}\right)$$

Using L_0^{-1} as a right preconditioner we come to the following equation:

$$(I - \delta L L_0^{-1}) < \widetilde{u} >= f, \quad \text{with} \quad u = L_0^{-1} < \widetilde{u} >$$

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3. The key point of any iterative technique is efficient implementation of the matrix-by-vector multiplication. The matrix we deal with comes from approximation of operators L_0^{-1} and δL. The latter one is pointwise multiplication, so let us deal with the first. In order perform Krylov's iterations one needs to compute the result of its action onto arbitrary 3D function, that is one should resolve the following Helmholtz equation:

$$\left(\Delta + (1+i\beta)\frac{\omega^2}{c_0^2(z)}\right)u_{tmp} = f_{input}(\vec{x})$$

After 2D Fourier transform with respect to lateral coordinates this equation is transformed into a two-parameter family of ordinary differential equations:

$$\frac{d^2U(z;k_x;k_y)}{dz^2} + \left((1+i\beta)\frac{\omega^2}{c_0^2(z)} - k_x^2 - k_y^2\right)U(z;k_x;k_y) = F(z;k_x;k_y) \quad (2)$$

To avoid artificial reflections at the bottom interface the transparent boundary condition (TBC) are introduced there. For the top one there is a choice - to consider it as a free surface or to apply there TBC as well.

In order to resolve the system (2) the 1D background velocity $c_0(z)$ is treated as a piecewise constant function. Then coefficients of ODE (2) may be considered as constant within each interval and its solution there is represented as a superposition of up- and downgoing waves:

$$U_j(z) = B_j \exp(-i \cdot K_j \cdot z) + C_j \exp(i \cdot K_j \cdot z) + G_j(z)$$

with G_j being some partial solution of heterogeneous ODE. Constants B_j and C_j are computed from conditions at the interaces:

$$[U]|_{z=z_j} = [\frac{dU}{dz}]|_{z=z_j} = 0 \quad (3)$$

which lead to a system of linear algebraic equations with tri-diagonal matrix. The desired solution is recovered from the computed coefficients in an evident manner. To come back to lateral coordinates, one needs to apply an inverse Fourier transform and finalize computation of the vector $u_{tmp}(\vec{x})$. The last step of this matrix-by-vector product is pointwise multiplication by the given 3D function:

$$\widetilde{u}_{output}(\vec{x}) = f_{input}(\vec{x}) - [\delta L](\vec{x}) \cdot u_{tmp}(\vec{x}) \quad (4)$$

4. The proposed approach was tested on the velocity model presented in Fig.1a (Gullfaks field, see ([3])). The wavefield was computed by the presented above iterative approach for Ricker source function with dominant frequency of 50 Hz. Computations were done for 601 temporal frequencies uniformly placed from 0.1 Hz to 90 Hz. Dependence of the number of iterations of the frequency one can see in Fig.1b. The model is discretized with with spatial steps $h_x = h_z = 5m$, that is 6 points per dominant wavelength. Snapshot corresponding to $t = 1.4s$ is presented in Fig.1c. As one can see, there is no visible manifestations of numerical dispersion.

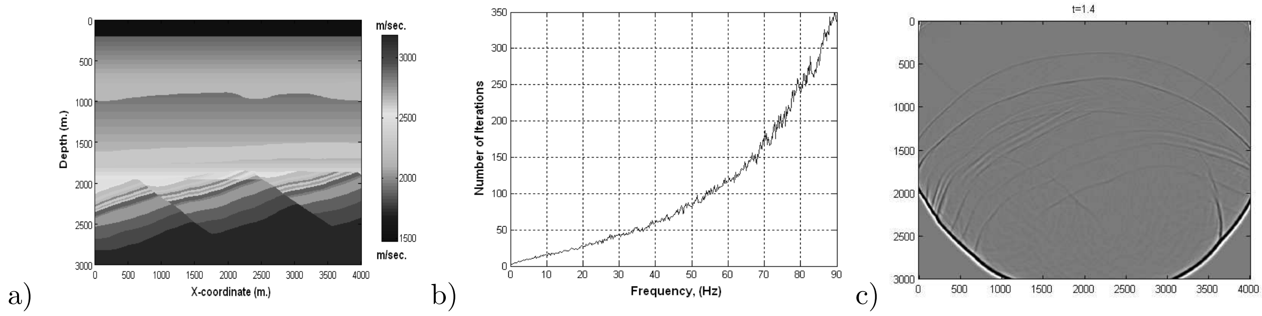

Figure 1: a) Gullfaks velocity model. b) Number of iterations vs frequency c) Snapshot for t=1.4 s

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