A Novel Aggregation Method based on Graph Matching for Algebraic MultiGrid Preconditioning of Sparse Linear Systems

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Multilevel techniques are very effective tools for preconditioning iterative Krylov methods in the solution of sparse linear systems; among them, Algebraic MultiGrid (AMG) are widely employed variants. In [2, 4] it is shown how parallel smoothed aggregation techniques can be used in combination with domain decomposition Schwarz preconditioners to obtain AMG preconditioners; the effectiveness of such a combination results from the fact that the use of coarse grids induces a higher coupling between the subdomains defined in the Schwarz framework.

Basically, in aggregation algorithms, the nodes of the graph associated with the matrix of the problem are coalesced to form supernodes, which in turn are the nodes of a coarser problem. By using these supernodes, AMG methods are provided with "ladders" to move from the fine level to the coarser ones and backward, i.e. with restriction and prolongation operators. Once we have these ingredients, we can use different "recipes" to combine the solution at the coarser levels with the solution and residual at the fine level, for computing corrections and improve the solution to the original linear system. The aggregation technique thus has a great impact on the effectiveness of an AMG-based preconditioner.

Aggregation techniques proposed in the past have been typically derived under the hypothesis of symmetric positive definiteness of the system matrix and much theoretical work exists on the optimality requirements for this case [5, 8]. Many commonly used aggregation techniques are based on greedy algorithms and do not fully exploit the information about the strength of connections between nodes provided by the original matrix. Furthermore, most of the theory developed in the literature is no longer applicable to the case of systems with unsymmetric matrices, where classical methods lose most of their efficiency.

In order to overcome these limitations, we decided to investigate the use of graph matching inside coarsening techniques applied to unsymmetric matrices; indeed the use of graph techniques in sparse matrix computations is well established, and with this work we provide yet another instance of this fruitful cooperation.

An effective coarsening has to represent well the smooth errors at the coarser level. Smooth errors are those which are not affected by the relaxation phase; they are usually called "slow to converge errors" or "algebraically smooth errors". It can be shown that algebraically smooth errors vary slowly in the direction of strong connections; as a result, an effective coarsening should aim at catching the strong couplings between variables, i.e. the large off-diagonal entries [7].

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			Classical Aggregation		MWM Aggregation		
Matrix	**Size**	**NNZ**	**# It**	**Flops**	**# MS**	**# It**	**Flops**
cdde1	961	4681	63	1.0e+07	3	24	3.9e+06
tub1000	1000	3996	unc.	-	2	17	2.3e+06
orsirr_1	1030	6858	1156	2.8e+08	3	937	2.3e+08
sherman4	1104	3786	11	1.5e+06	3	12	1.6e+06
epb0	1794	7764	764	1.9e+08	4	1478	3.9e+08
CAG_mat1916	1916	195985	995	5.8e+09	5	1017	5.9e+09
bwm2000	2000	7996	unc.	-	3	37	1.1e+07
orsreg_1	2205	14133	542	3e+08	3	464	2.6e+08
add20	2395	13151	1740	1.5e+09	3	1728	1.5e+09
wang2	2903	19093	43	3.4e+07	3	23	1.8e+07
pde2961	2961	14585	10	5.4e+06	3	13	7.2e+06
raefsky2	3242	293551	unc.	-	5	871	7.6e+09
shermanACa	3432	25220	208	3.5e+08	3	182	2.3e+08

Table 1: Preliminary results with maximum weighted matching aggregation.

This work stems from the idea that a maximum weighted matching (MWM) performed on the matrix adjacency graph could be effective at maximizing the amount of strong couplings captured by the aggregation process. A matching of the matrix graph provides a trivial way of forming aggregates of size two (plus a negligible number of aggregates of size one due to the presence of unmatched nodes). As a result, the cost of computing the coarse-level correction may be too high; in order to cope with this issue, multiple matching-based aggregation steps can be applied until aggregates of the desired size are built.

The proposed approach was experimented in an AMG solver prototype developed from scratch in the Matlab computing environment. This software implements a basic Algebraic MultiGrid solver with a V-cycle correction type, different kinds of smoothers, classical [1, 8] or MWM aggregations, and Galerkin or Petrov-Galerkin approaches [6] for building the coarse matrices. The code relies on the MC64 package [3] for computing the maximum weighted matching of a graph. In the case of MWM aggregation, the code can be tuned, through specific parameters, to adopt different strategies for handling the presence of unmatched nodes as well as for controlling the size of the aggregates. This code represents a robust environment for validating the proposed approach to aggregation and to conduct preliminary experiments on sparse linear systems produced by real-world applications.

Table 1 reports preliminary results obtained on a number of unsymmetric matrices from the University of Florida Sparse Matrix Collection[7]. All the tests were done using both the MWM and classical aggregation as coarsening algorithms, the Petrov-Galerkin approach with local damping for the construction of the coarse matrix, and the Gauss-Seidel pre/post-smoother. The number of levels was set to two and the iterations were stopped when either the relative

[7]http://www.cise.ufl.edu/research/sparse/matrices/

residual was less than 10^{-6} or a maximum number of iterations, fixed to 5000, was exceeded. In the case of MWM aggregation, multiple matching steps were performed until the size of the coarse problem was comparable to that obtained with the basic aggregation; the number of matching steps is reported, for each matrix, in column six of Table 1.

As the size of the aggregates for both MWM and classical aggregation is comparable, the cost per iteration is similar and, thus, a reduction in the number of iterations yields an equivalent reduction in the number of floating-point operations. The table shows that the proposed approach has the potential to deliver a robust and efficient aggregation method for solving and preconditioning unsymmetric systems through AMG methods. Indeed, for most of the test problems the matching-based aggregation results in a faster convergence than the classical aggregation; for the remaining problems it leads to a comparable (e.g., sherman4 or add20) or slower (e.g., epb0) AMG method. Ongoing work aims at identifying the numerical and structural properties of a matrix that make it suitable for MWM aggregation.

Our plan includes integrating and experimenting the novel aggregation algorithms in the algebraic multilevel Schwarz parallel preconditioning framework provided by MLD2P4 [2], and thus making them freely available.

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