Generalized Filtering Decomposition

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In this talk we discuss a new preconditioning method for solving a large sparse linear systems of equations arising from the discretization of a partial differential equation (PDE) by a finite difference or finite volume method on an unstructured grid in two or three dimensions. The preconditioner is based on an incomplete block LDU decomposition and is suitable for parallel computation.

The problem of preconditioning has already been extensively studied, see for example [3], [5], [2] for descriptions of algorithms and comparisons. In particular, one class of methods which have proved particularly successful are the algebraic multigrid methods, see for example [4], [6], [7]. They are very successful for scalar PDEs and for a limited number of processors. This motivates research on iterative solvers for systems of PDEs and/or larger number of processors.

Our work is in the context of filtering preconditioners, that for a given filtering vector t satisfy the relation $Mt = At$, where A is the input matrix and M is the preconditioner. Previous work on tangential filtering preconditioners [1] has focused on developing preconditioners for matrices arising from the discretization of a partial differential equation (PDE) on a structured grid in two or three dimensions. The tangential filtering precondioner has been used in combination with ILU0.

In this talk we present a block preconditioner that satisfies the filtering property and that does not impose any particular structure on the input matrix. It can be seen as a generalization for unstructured grids of the preconditioner presented in [1] for block tridiagonal matrices. In contrast to the preconditioner presented in [1] that has been shown to be efficient in combination with ILU0, the block preconditioner presented here is efficient as a stand-alone preconditioner.

Consider a matrix A of size $n \times n$ partitioned into a block matrix of size $N \times N$ with square diagonal blocks (not necessarily of a same size).

$$A = \begin{pmatrix} A_{11} & \dots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \dots & A_{NN} \end{pmatrix},$$

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An exact block LDU factorization of A is written as

$$A = \begin{pmatrix} D_{11} & & & \\ L_{21} & D_{22} & & \\ \vdots & \ddots & \ddots & \\ L_{N1} & \dots & L_{N,N-1} & D_{NN} \end{pmatrix} \begin{pmatrix} D_{11}^{-1} & & & \\ & D_{22}^{-1} & & \\ & & \ddots & \\ & & & D_{NN}^{-1} \end{pmatrix} \begin{pmatrix} D_{11} & U_{12} & \dots & U_{1N} \\ & \ddots & \ddots & \vdots \\ & & D_{N-1,N-1} & U_{N-1,N} \\ & & & D_{NN} \end{pmatrix}, \quad (1)$$

where $D_{ii}, i = 1 \dots N$ are square invertible matrices of size $b_i \times b_i$ with $b_i < n$. In practice, even if the matrix A is very sparse, the factors L, D, U can be much denser. In particular, multiplications that involve the inverse matrices D_{kk}^{-1} can introduce a lot of fill-in in the factors.

In the proposed block filtering preconditioner, the inverse of the diagonal blocks $D_{kk}^{-1}, k = 1 \dots n$ is approximated by a sparse matrix such that the preconditioner M stays sparse. In addition M satisfies the right filtering condition $(M - A)t = 0$, where t is a filtering vector. The choice of the vector can be made similar to deflation techniques. However, in practice we find that a vector of all ones leads to good convergence results.

In our tests we use a reordering of the matrix that is suitable for parallel implementations. This reordering, known as nested dissection, allows a parallel implementation of the construction of the preconditioner, as well as of the iterative process.

We present numerical results for two classes of problems. The first class is composed of matrices arising from the discretization of scalar PDEs on structured grids. The second class is composed of matrices from several real applications as reservoir modelling, nuclear waste disposal, etc. These results show that the preconditioner is very efficient for the matrices in our test set, which are of medium size. In our future work we focus on a parallel implementation of the preconditioner, that will allow us to test it on larger matrices.

References

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